BIFILAR GNOMONICS

FREDERICK W. SAWYER, III
137 Mallard Drive, East Hartford, Connecticut 06118, USA

Bifilar gnomonics is the study of one of the few truly twentieth-century types of sundials. The bifilar sundial was invented in 1922 by Professor Hugo Michnik[a]*, an Oberlehrer at the Kgl. Gymnasium in Beuthen, Upper Silesia, Germany (now Bytom, Poland). The dial has the advantage of having equiangular hour-lines: the lines all intersect in a single point and the angles between successive hour-lines are uniformly 15°. The usual sort of sundial, based on a gnomonic projection, does not display this uniformity in the placement of the hour-lines. Besides making the dial easier to construct, the equiangular feature permits a very easy daily adjustment of the dial face to give a direct reading of standard clock time rather than local apparent time: once the difference between the two times is determined for a given date and location, the required adjustment amounts to a simple rotation of the dial face through an angle whose size is proportional to that difference. Unlike other horizontal equiangular dials[b], this one does not require an additional daily adjustment of the gnomon; indeed, in the case of the bifilar sundial, there is no gnomon to adjust. Instead of the usual shadow-casting device the dial uses two horizontal threads (hence the term 'bifilar') suspended at right angles to one another at appropriate heights above the dial face. Although the threads do not intersect, their shadows do; and it is their intersection which indicates the time.

The purpose of the present paper is to elaborate on Michnik's somewhat condensed treatment of the theory of the dial, thus making it more accessible to modern diallers. Besides reproducing Michnik's results and presenting a simplified justification of the construction, we will consider dials on arbitrary planes, a combination of dials which will indicate the time whenever the Sun is above the horizon, and general geometrical procedures for drawing Babylonian, Italian and sidereal hour-lines.

Construction

In order to construct a bifilar sundial for a latitude north of the equator, begin with a circle of equiangular hour-lines: hours are marked clockwise around the circle at 15° intervals; each degree corresponds to 4 minutes of time. Suppose perpendicular lines NQS and EQW (figure 1) are drawn

*The letters refer to notes at the end of this paper.

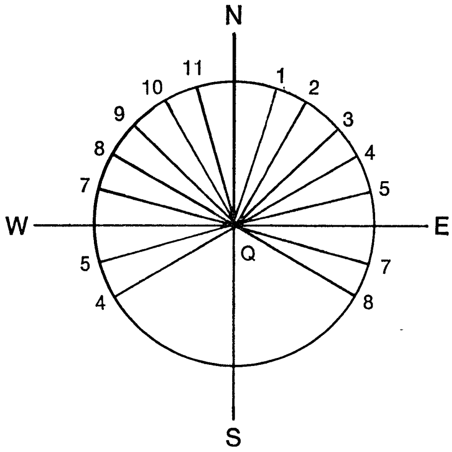

FIGURE 1.

on a horizontal base; the hour-circle should be attached to this base at its centre Q so that it can rotate freely. The line SN will represent the meridian; the direction from Q to N is north. Let O be a point on QN such that the segment QO has unit length. Suspend a thread horizontally above O in the east–west direction with height tan ϕ, where ϕ is the geographical latitude at which the dial is to be used. Suspend a second thread horizontally above O in the north–south direction with height sec ϕ. To set the dial up, place it on a horizontal surface with the threads in the directions indicated, using true geographic north. Rotate the hour-circle so that the noon-line lies along the meridian QN. The intersection of the threads' shadows on the dial face will register local apparent time. Alternatively, if the hour-line corresponding to the (clock) time at which the Sun crosses the meridian on any given day is made to lie along QN, then the dial will agree with the clock throughout the day.

There are a number of ways to determine the direction of true north, the easiest of which uses the dial itself with an accurate clock. After rotating the dial face to indicate Standard (or Summer) Time, rotate the entire dial base until the correct time is read; the dial will then be correctly oriented and will continue to give correct readings throughout the year with only the minor modifications already noted.

Local noon, the time of the Sun's crossing the meridian for any given day and location, may be determined by reference to the Equation of Time (a table which is available in most almanacs). To determine the Standard Time of local noon, the appropriate entry in the table is to be added algebraically to the base time which is noon plus $4(\lambda-\lambda')$ minutes, where λ is the longitude, expressed in degrees, of the dial's location and λ' is the standard meridian of the dial's time zone. During Summer Time, of course, an additional hour must be added. Alternatively, local noon may be determined simply as the time midway between local sunrise and sunset.

For a dial south of the equator the hours are marked counterclockwise around the circle. The point O should be selected on the southern line segment QS which now corresponds to the noon-line. The heights of the threads above O are the same as they would be for a dial at the corresponding latitude above the equator[c].

Justification

Suppose that the threads on the dial are attached to a vertical rod with base at point O. In figure 2, the shadow of this rod is OR; lines UP (parallel to QE) and RP (parallel to QN) are the shadows of the threads, and P is therefore the intersection of the shadows. We will need the following identities:

$$\cos h \cos A = \sin\phi \cos\delta \cos t - \cos\phi \sin\delta \qquad (1)$$

$$\cos h \sin A = \cos\delta \sin t \qquad (2)$$

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t \qquad (3)$$

where t = solar hour-angle; A = solar azimuth; h = solar altitude; δ = solar declination.

In order to justify the equiangular construction of the dial described above, we need to establish that the angle w between the meridian line QN and the hour-line QP is equal to the Sun's hour-angle t.

$$\overline{OU} = \tan\phi \cot h \qquad \overline{OR} = \sec\phi \cot h$$

$$\begin{aligned} \tan w = \overline{PV}/\overline{QV} &= \overline{RT}/(1+\overline{OV}) \\ &= \sec\phi \cot h \sin A/(1+\tan\phi \cot h \cos A) \\ &= \cos h \sin A/(\cos\phi \sin h + \sin\phi \cos h \cos A) \\ &= \cos\delta \sin t/(\cos^2\phi \cos\delta \cos t + \sin^2\phi \cos\delta \cos t) \\ &= \tan t \end{aligned}$$

Thus, $w = t$ and the equiangular bifilar sundial is justified. The case for a southern dial is similar.

Optimum radius

One drawback of the bifilar sundial is that it does not indicate the hour for the entire time that the Sun is above the horizon; as the Sun rises or sets, the distance $\overline{QP}$ from the centre to the point of intersection of the shadows becomes larger than the radius of the dial. Referring to figure 2, the distance $\overline{QP}$ is determined as follows:

$$\begin{aligned} \overline{QP} = \overline{PV}/\sin w &= \sec\phi \cot h \sin A \operatorname{cosec} t \\ &= \sec\phi \operatorname{cosec} h \cos\delta && \text{by (2)} \\ &= \sec\phi/(\sin\phi \tan\delta + \cos\phi \cos t) && \text{by (3)} \end{aligned}$$

Thus the latest hour-angle t_f which can be read on a dial of radius ρ (in units equal to the distance $\overline{QO}$) for a given declination δ is determined by setting $\overline{QP} = \rho$ and solving for t:

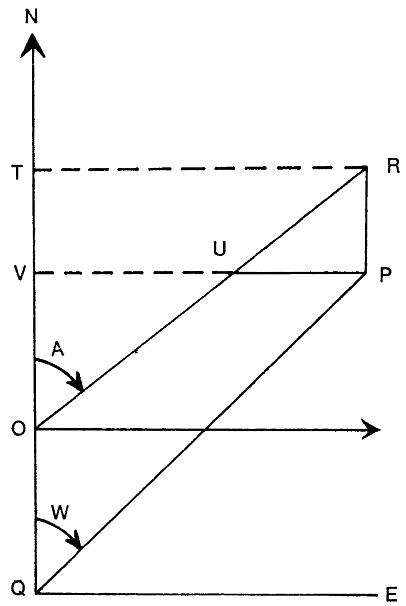

FIGURE 2.

$$t_f = \cos^{-1}[(1-\rho \cos\phi \sin\phi \tan\delta)/(\rho \cos^2\phi)] \ .$$

The hour-angle t_s of sunset for a given day and latitude is

$$t_s = \cos^{-1}[-\tan\phi \tan\delta] \ .$$

As a guide for the selection of a convenient radius, values for the ratio t_f/t_s of these hour-angles appear in Tables I and II for latitudes 40°N and 51° 30 N, respectively. A procedure for constructing a combination of dials which indicates the time whenever the Sun is above the horizon will be discussed below.

TABLE I

VALUES OF THE RATIO t_f/t_s FOR LATITUDE 40°N

Radius	*Summer Solstice*	*Equinox*	*Winter Solstice*
3	0·703	0·615	0·310
4	0·776	0·720	0·551
5	0·820	0·779	0·658
6	0·849	0·817	0·723
7	0·870	0·843	0·766
8	0·886	0·863	0·798
9	0·899	0·879	0·822
10	0·909	0·891	0·840

Dials for arbitrary Planes

Although we have been considering dials which lie on a horizontal plane, it is possible to construct a bifilar dial for any given plane. Suppose we draw a north–south line N_1QS_1 and an east–west line E_1QW_1 on a horizontal plane and then rotate the plane through an angle i about E_1W_1 so that the line segment QN_1 is above the horizon. Suppose further that the plane is then rotated clockwise about the vertical line QZ (where Z is the zenith point directly overhead) through an angle d. As a result of these rotations the plane is said to have inclination i and declination d. In order to construct a dial on an arbitrary plane the only information needed in addition to the dial's latitude is the inclination and declination of the plane on which it is to rest. Throughout the following, the line QN_1 (figure 3) on a given plane will be understood to be the north ray of the line N_1QS_1 which is obtained in the manner just described: beginning as a horizontal meridian line and then being rotated so that it lies on the given plane.

TABLE II

VALUES OF THE RATIO t_f/t_s FOR LATITUDE 51°30N

Radius	*Summer Solstice*	*Equinox*	*Winter Solstice*
3	0·582	0·341	–
4	0·685	0·554	–
5	0·745	0·655	–
6	0·785	0·717	0·225
7	0·814	0·760	0·421
8	0·836	0·791	0·523
9	0·853	0·815	0·592
10	0·867	0·834	0·642

Now to construct an equiangular bifilar sundial for latitude ϕ on a plane with inclination i and declination d, begin by selecting a point Q on the plane and draw the line segment QN_1. Draw the perpendicular lines NQS and EQW (figure 3) with the angle θ from QN_1 to QN determined as follows:

$$\cos\theta = \frac{\tan\phi\sin i+\cos d\cos i}{((\tan\phi\sin i+\cos d\cos i)^2+\sin^2 d)^{\frac{1}{2}}}$$

where the direction from QN_1 to QN is counterclockwise if d is positive and clockwise if d is negative. QN is the intersection of the given plane with the plane determined by its pole and the celestial axis.

Alternatively, the lines NQS and EQW may be determined by first drawing the north–south line QN_2 (figure 3); *i.e.* the line which is the intersection of the given plane and the meridian plane. The angle θ' from QN_2 to QN is determined as follows:

$$\tan\theta' = \frac{\sin\phi\cos i-\cos\phi\sin i\cos d}{\sin\phi\cot d+\cos\phi\cot i\operatorname{cosec} d}\ .$$

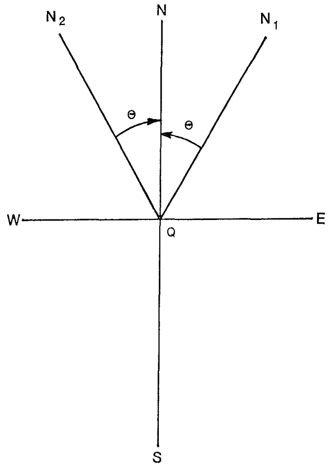

FIGURE 3. (The angle N_2QN should be θ'.)

Now at latitude ϕ and longitude λ a plane with inclination i and declination d is parallel to a horizontal plane at latitude α and longitude $(\lambda+t')$, where longitude is understood to increase to the west, and

$$\sin \alpha = \sin \phi \cos i - \cos \phi \sin i \cos d \tag{4}$$

$$\sin t' = \sin i \sin d \sec \alpha$$

$$\cot t' = \sin \phi \cot d + \cos \phi \cot i \operatorname{cosec} d \tag{5}$$

Moreover, if north–south and east–west lines were drawn on that horizontal plane, they would be parallel respectively to the lines NS and EW which have already been drawn on the given plane. We can therefore simply construct the dial over these lines as before, but in doing so we must use α as our latitude as though the plane were displaced to the location in which it would be horizontal.

The dial is now complete, but it is important to note that the point N (or, if $\alpha < 0°$, the point S) no longer corresponds to local noon but rather to the hour-angle t' (*i.e.* the time past noon expressed in degrees, where 1° equals 4 minutes of time).

As examples, suppose first that we wish to construct at latitude 40°N a vertical sundial which faces directly south. For such a dial we have $i = 90°$ and $d = \theta = \theta' = 0°$; the line NS is vertical. Further calculation yields $\alpha = -50°$ and $t' = 0°$. The dial to be constructed may thus be treated as though it were a horizontal dial at latitude 50°S.

Since the distance from the shadows' intersection to the centre of the horizontal dial is generally greater when the Sun is low above the horizon, a vertical dial used at a northern latitude will be easier to read during the spring and summer than will a horizontal one: during these seasons the Sun is low in the sky at the southern latitude whose horizontal plane is parallel to the vertical dial. However, although such a dial would be

TABLE III

USABLE PERIOD OF DIALS WITH RADIUS 2·95 AT LATITUDE 40°N

	East	*Horizontal*	*West*
Summer Solstice	4:35–10:41	6:49–5:11	1:19–7:25
Equinoxes	6:00–10:41	8:21–3:39	1:19–6:00
Winter Solstice	7:25–10:41	10:41–1:19	1:19–4:35

easier to read, it would indicate time during a smaller portion of the daylight.

Now suppose that at latitude 40°N we want a vertical dial facing directly east, so that $i = 90°$, $d = -90°$ and $\theta = +50°$, where the positive value for θ indicates a clockwise measurement from QN_1 to QN. The line QN_1 is again vertical, so QN is parallel to the celestial axis. Since $\alpha = 0°$, the dial may be constructed as though it were in either hemisphere, the different choices determining the placement of the point O and whether the hours are marked in a clockwise or counterclockwise sense. Because $t' = -90°$, the line QN (or QS, depending on the placement of O) is now the hour-line for 6 am local apparent time. The resulting dial is equivalent to the classical vertical direct east dial.

Direct east and west dials may obviously be used to read early morning and late afternoon hours, respectively. A combination of these dials with a horizontal one, all of them being displayed, for example, on the faces of a cube, will indicate the time whenever the Sun is above the horizon, provided their radii are chosen appropriately. Thus, if all three dials have the radius 2·95 (where the line QO has unit length) at latitude 40°N, the time during which each dial is usable is given in Table III for selected dates.

The equations for determining appropriate values for the radius ρ are complex. On the simplifying assumption that the dials share a common radius and are either horizontal or vertical, the value chosen for ρ must be such that conditions (6) and (7) hold, where $\epsilon = 23;26°$.

$$\cos^{-1}[(1+|\rho\cos\phi\sin\phi\tan\epsilon|)/(\rho\cos^2\phi)]+\cos^{-1}[1/\rho] \geqslant 90° \qquad (6)$$

$$\cos^{-1}[-|\tan\phi|\tan\delta]-\cos^{-1}[1/\rho] \leqslant 90° \qquad (7)$$

The first condition guarantees that the dial combination records the time whenever the Sun is above the horizon between 6 am and 6 pm; the second imposes a similar guarantee for earlier risings and later settings of the Sun. It can be shown that whenever condition (6) is satisfied, so is condition (7). Thus the minimum—and, from the point of view of readability, the best—value for ρ at latitude ϕ is obtained by considering the case of equality in (6) and solving for ρ. This yields the following equation:

$$\rho = K + \sqrt{K^2 + (\sec^2\phi + \cos^2\phi)/(1 - \sin^2\phi\sec^2\epsilon)}$$

where $K = |\tan\phi\tan\epsilon|/(1-\sin^2\phi\sec^2\epsilon)$.

At latitude 40°N, this equation gives the value $\rho = 2{\cdot}95$; at latitude 51° 30 N we have $\rho = 5{\cdot}86$. Additional values for various latitudes are given in Table IV.

Throughout the following, the equations to be given apply to horizontal dials; however, appropriate adjustments may be made to adapt them to dials on arbitrary planes.

TABLE IV

OPTIMUM RADII FOR BIFILAR CUBE DIALS

ϕ	ρ	ϕ	ρ	ϕ	ρ
30°	2·11	40°	2·95	50°	5·21
31°	2·17	41°	3·08	51°	5·63
32°	2·23	42°	3·23	52°	6·11
33°	2·30	43°	3·39	53°	6·68
34°	2·37	44°	3·57	54°	7·35
35°	2·45	45°	3·77	55°	8·14
36°	2·53	46°	3·99	56°	9·11
37°	2·62	47°	4·24	57°	10·29
38°	2·72	48°	4·52	58°	11·77
39°	2·83	49°	4·84	59°	13·67

General theory

The justification given earlier for equiangular bifilar sundials presupposed that the heights of the two threads were known. The general development to be given here makes no such supposition and, as a result, it will demonstrate not only that the values already given are the only ones which produce an equiangular dial, but also that a variety of different (non-equiangular) dials may be obtained by appropriate changes in the heights.

Consider a rectangular co-ordinate system with origin at point O and such that the x and y-axes are directed east and north, respectively (see figure 2). Suppose that horizontal threads are suspended above O, one along the y-axis at height g_1 and the other along the x-axis at height g_2. As before, suppose that a solid vertical rod has its base at O; then P is the intersection of the shadows of the threads, and

$$\overline{\mathrm{OU}} = g_2 \cot h \qquad\qquad \overline{\mathrm{OR}} = g_1 \cot h$$

The co-ordinates of P are

$$x = g_1 \cot h \sin A \tag{8}$$

$$y = g_2 \cot h \cos A \tag{9}$$

Using the identities (1)–(3) above, these equations may be transformed into the following:

$$x = g_1 \sin t/(\sin \phi \tan \delta + \cos \phi \cos t) \quad (10)$$

$$y = g_2 (\sin \phi \cos t - \cos \phi \tan \delta)/(\sin \phi \tan \delta + \cos \phi \cos t) \quad (11)$$

Solving these equations for $\tan \delta$ and then eliminating this parameter yields an equation for the hour-lines[d], dependent only on ϕ and t:

$$g_2\, x \cot t - g_1\, y \sin \phi = g_1 g_2 \cos \phi$$

Since this equation is linear in x and y, the sundial has straight hour-lines. All of these lines intersect in one point Q, the y-intercept (obtained by setting $x = 0$).

$$x = 0 \quad \rightarrow \quad y = -g_2 \cot \phi \qquad \overline{\mathrm{QO}} = g_2 \cot \phi$$

$$y = 0 \quad \rightarrow \quad x = g_1 \cos \phi \tan t \qquad \overline{\mathrm{OV}} = g_1 \cos \phi \tan t$$

Suppose we choose our unit length so that $\overline{\mathrm{QO}} = 1$; then $g_2 = \tan \phi$. We also have $\tan w = \overline{\mathrm{OV}}/\overline{\mathrm{QO}} = k \tan t$, where $k = g_1 \cos \phi$. In order to obtain an equiangular dial ($w = t$) we must set $k = 1$; we then have $g_1 = \sec \phi$.

However, suppose now that we relinquish the equiangular requirement. Then a variety of dials may be constructed corresponding to different values for k. As an example, suppose we have a usual sort of horizontal (hour-arc) dial constructed for latitude ϕ'. If we identify the meridian line of this dial with QON and the centre of its hour-lines with the point Q, then any text on gnomonics will tell us that

$$\tan w' = \sin \phi' \tan t$$

We can therefore adapt the dial for latitude ϕ and still keep it horizontal by removing the gnomon and erecting a horizontal east–west thread above point O at height $\tan \phi$ and a similar north–south thread at height $\sin \phi'/\cos \phi$; that is, simply set $k = \sin \phi$. If $\phi = \phi'$, then $g_1 = g_2$ and we have an ordinary horizontal dial.

Day curves

On many sundials there are curves drawn to trace the path of the shadow of a particular point on the gnomon for selected dates. While this practice would serve no purpose in an equiangular bifilar sundial with a rotating dial face (unless of course the face were transparent and the curves were drawn on the non-rotating base), nevertheless one may determine exactly what these curves will be in general. The curve we will consider is traced by the intersection of the threads' shadows.

From equation (11) we have

$$\cos t = \tan \delta\, (g_2 \cos \phi + y \sin \phi)/(g_2 \sin \phi - y \cos \phi)$$

Using this result in equation (10) yields

$$\sin t = g_2\, x \tan \delta/(g_1 g_2 \sin \phi - g_1\, y \cos \phi)$$

Since $\sin^2 t + \cos^2 t = 1$, we have

$$\tan^2 \delta \, (g_2^2 \, x^2 + g_1^2 \, (g_2 \cos \phi + y \sin \phi)^2) = g_1^2 \, (g_2 \sin \phi - y \cos \phi)^2$$

Finally, by means of the identity $\cos^2 \phi = 1 - \sin^2 \phi$, we have the following equation of the day curve[e], dependent only on ϕ and δ,

$$\sin^2 \delta \, (g_2^2 \, x^2 + g_1^2 \, y^2 + g_1^2 g_2^2) = g_1^2 \, (g_2 \sin \phi - y \cos \phi)^2$$

Note, however, that this equation is based on considering O as the origin. If Q, the centre of the equiangular dial, were used as the origin, the variable y would have to be replaced by $(y - g_2 \cot \phi)$.

On the equinoxes, when $\delta = 0°$, the day curve becomes a straight line from west to east. More generally, the form of the curve on any given day may be determined from Table V.

For a horizontal equiangular dial (constructed say for a northern latitude), it is probably easier to trace the day curves by using polar co-ordinates with Q as origin, since at any time t the point P has the polar co-ordinates $(t, \overline{QP})$.

TABLE V

THE FORM OF THE DAY CURVE ON SELECTED DATES

Solar Declination	*Day Curve*
$\delta > 90 - \phi$	ellipse
$\delta = 90 - \phi$	parabola
$\delta < 90 - \phi$	hyperbola

Altitude and azimuth curves

In addition to day curves, a dial may be furnished with curves of equal solar altitude or azimuth. Thus, for example, on a bifilar sundial with curves for altitude $h = 30°$ or azimuth $A = 40°$, the intersection of the shadows will lie on one of these curves exactly when the Sun has altitude 30° or azimuth 40°, respectively.

The equation for altitude curves is obtained by eliminating the parameter A from equations (8) and (9) to obtain

$$(x \tan h/g_1)^2 + (y \tan h/g_2)^2 = 1 \; .$$

The resulting curve is an ellipse.

Using equations (8) and (9) again, but this time eliminating the parameter h, we have

$$y = (g_2/g_1) \, x \cot A$$

The azimuth curve is thus a straight line through the origin O. The line corresponding to solar azimuth A itself has an azimuth A', where $\cot A'$ is the slope (y/x) of the line. For an equiangular dial the two angles are therefore related to each other as follows:

$$\cot A' = \sin \phi \cot A$$

Babylonian and Italian hour-lines

As a general rule in gnomonics, the zero-point for determining the hour-angle of the Sun for any particular location is local noon, when the Sun is on the meridian. However, this need not be the case; we will consider here two other well-known systems: the Babylonian, which reckons time from the preceding sunrise, and the Italian, which reckons time from the preceding sunset. Consider a sundial which has both Babylonian and Italian hour-lines but on which the Italian lines are numbered in reverse, so that sunset is 0 hours, while one hour before sunset is 1 hour. Such a dial may be used to determine at any given time how many hours have elapsed since sunrise (the Babylonian hour b), how many hours remain until sunset (the reversed Italian hour i), how many hours of daylight there are on the given day $(b+i)$, and the local apparent time $\frac{1}{2}(b-i)$, where a zero value for t denotes local noon.

If we let T be the time from sunrise to noon on any given day and let b be the Babylonian hour, then

$$t = b-T \qquad \text{and} \qquad \cos T = -\tan\phi\tan\delta\ .$$

Substituting these equations into the equations (10) and (11) for the co-ordinates of the point P yields:

$$x = \frac{g_1(\sin b - \cos b \tan T)}{(\cos\phi\cos b + \cos\phi\sin b\tan T - \cos\phi)}$$

$$y = \frac{g_2(\sin\phi\cos b + \sin\phi\sin b\tan T + \cos\phi\cot\phi)}{(\cos\phi\cos b + \cos\phi\sin b\tan T - \cos\phi)}$$

Solving these equations for, and then eliminating, $\tan T$ results in the equation for Babylonian hour-lines:

$$g_1 y\sin\phi\cos\phi(1-\cos b) - g_2 x\sin b\cos\phi = g_1 g_2(\sin^2\phi + \cos^2\phi\cos b) \quad (12)$$

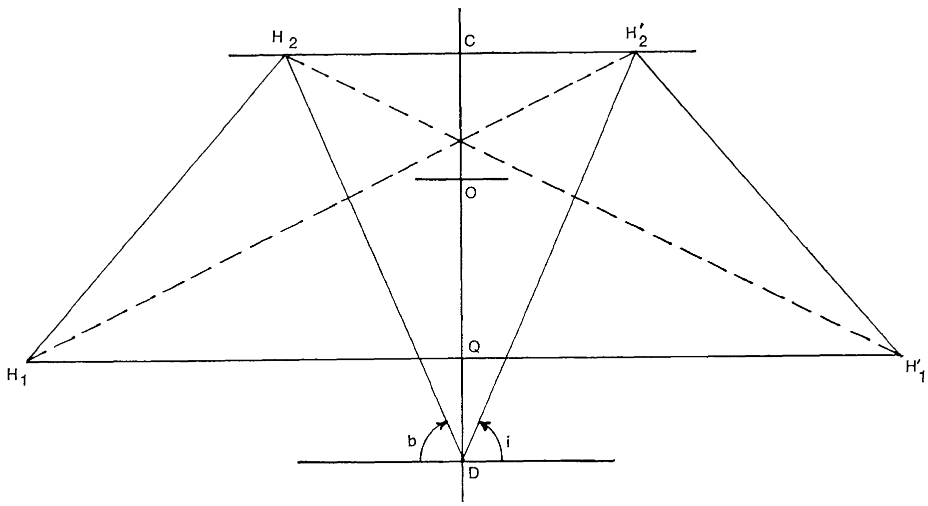

FIGURE 4.

TABLE VI

CO-ORDINATES OF SELECTED POINTS FOR THE CONSTRUCTION OF BABYLONIAN AND ITALIAN HOUR-LINES

Point	*General Case* X	*General Case* Y	*Special Case* X	*Special Case* Y
Q	0	$-g_2 \cot \phi$	0	-1
C	0	$g_2 \tan \phi$	0	$\tan^2 \phi$
D	0	$g_2 \tan \phi - g_1 \sec \phi$	0	-1

By similar reasoning for the (reverse-numbered) Italian hours i: T may also be viewed as the time from noon to sunset; so $t = T-i$ and the resulting equation is:

$$g_1 y \sin \phi \cos \phi(1-\cos i)+g_2 x \sin i \cos \phi = g_1 g_2 (\sin^2 \phi+\cos^2 \phi \cos i) \quad (13)$$

Both families of hour-lines are linear; however, neither the Babylonian nor the Italian hour-lines have a common point of intersection, as is the case for the usual astronomical hour-lines. The angles β and ι that they respectively make with the line QON are as follows:

$$\tan \beta = (g_1/g_2) \sin \phi \tan \tfrac{1}{2}b \qquad \tan \iota = -(g_1/g_2) \sin \phi \tan \tfrac{1}{2}i$$

Thus, if $g_2 = g_1 \sin \phi$, the hour-lines are equiangular in the sense that β and ι are proportional to b and i, respectively.

Michnik gives a graphic means of drawing these hour-lines for special values of g_1 and g_2; it may be generalized as follows. In figure 4 let O be the origin as before. Table VI lists the co-ordinates of the points Q, C and D in the general case and in the special case in which $g_2 = \tan \phi$ and $g_1 = \sec \phi$. Construct lines CH_2 and QH_1 perpendicular to QC and let H_2 be the point of intersection of line DH_2 with line CH_2, where the angle $CDH_2 = b-90°$. Let point H_1 be such that $\overline{QH_1} = \overline{DH_2}$. The Babylonian hour-line for hour b is H_1H_2.

A similar construction, symmetric about the line QC and yielding points H_1' and H_2' works for Italian hour-lines.

To justify these constructions it suffices to show that points H_1 and H_2 as well as the symmetrically placed points H_1' and H_2' lie on the respective hour-lines. From the manner of selecting the points we have:

$$H_1 = (-g_1 \sec \phi \operatorname{cosec} b,\ -g_2 \cot \phi)$$
$$H_2 = (-g_1 \sec \phi \cot b,\ g_2 \tan \phi)$$
$$H_1' = (g_1 \sec \phi \operatorname{cosec} i,\ -g_2 \cot \phi)$$
$$H_2' = (g_1 \sec \phi \cot i,\ g_2 \tan \phi)$$

It is now a simple matter to verify that the co-ordinates of these points satisfy the equations (12) and (13) for the appropriate hour-lines.

It should also be noted that each of the four hour-lines obtainable from these points may be viewed as either Babylonian or Italian (see

Table VII). Although there are no lines corresponding to sunrise ($b = 0°$) or to sunset ($i = 0°$), the same line, given by the following equation, represents the twelfth hours after sunrise and before sunset:

$$y = g_2 (\tan \phi - \cot \phi)/2$$

Figure 5 displays Babylonian hour-lines, numbered clockwise, and Italian hour-lines, numbered counterclockwise, for a bifilar dial at latitude 40°N with $g_1 = \sec \phi$ and $g_2 = \tan \phi$.

TABLE VII

HOUR-LINES OBTAINABLE FROM A SINGLE CONSTRUCTION

Line	*Babylonian Hour*	*Italian Hour*
H_1H_2	b	$360° - b$
$H_1H'_2$	$180° - b$	$180° + b$
H'_1H_2	$180° + b$	$180° - b$
$H'_1H'_2$	$360° - b$	b

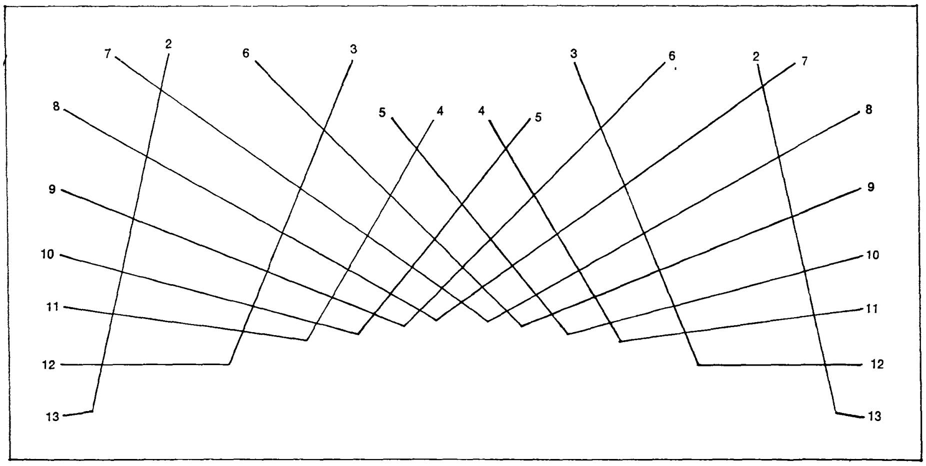

FIGURE 5. Babylonian and Italian hour-lines for $\phi = 40°N$.

Sidereal hour-lines

Finally[f], we will consider one more system of indicating time: sidereal hours. It is a fact well known to diallers that the length of the solar day, defined as the time between successive meridian-crossings by the Sun, is not constant. Because the Sun appears to have a non-uniform motion of its own along the ecliptic with respect to the fixed stars, the so-called Equation of Time must be added to the sundial's reading to obtain a result which increases uniformly. The chief advantage of an equiangular dial is that it affords an easy means of making the desired correction in the reading.

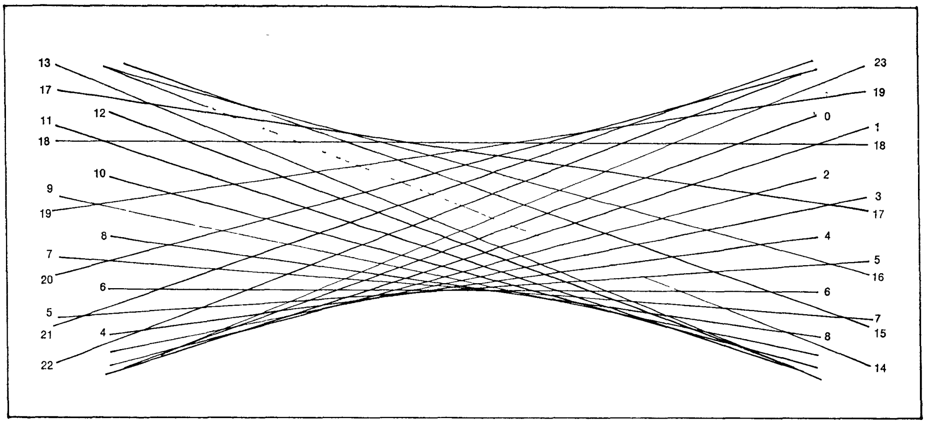

FIGURE 6. Sidereal hour-lines for $\phi = 40°$N.

Suppose, however, that the day is now defined as the interval between successive crossings of the meridian by a fixed point on the ecliptic—in particular, by the vernal point, *i.e.* the Sun's position on the day of the vernal equinox, which corresponds to the intersection of the ecliptic and celestial equator. This so-called sidereal day is of interest primarily in astronomical contexts; its length is constant and, on the average, is approximately 3 minutes 56 seconds shorter than a solar day. By definition, the local sidereal hour θ is the hour-angle of the vernal point. For any astronomical body, and in particular for the Sun, θ is equal to the body's hour-angle t (the angle west of the meridian measured along the equator) plus its right ascension α (the angle east of the vernal point measured along the equator).

To obtain an equation for sidereal hour-lines we begin with the equations relating θ to the Sun's hour-angle, right ascension and declination.

$$t = \theta - \alpha \qquad \tan \delta = \tan \epsilon \sin \alpha \qquad (\epsilon = 23;26°)$$

At this point, we proceed as before; using these equations for substitutions in (10) and (11), we obtain:

$$x = \frac{g_1 (\sin \theta - \cos \theta \tan \alpha)}{(\sin \phi \tan \epsilon + \cos \phi \sin \theta) \tan \alpha + \cos \phi \cos \theta}$$

$$y = \frac{g_2 ((\sin \phi \sin \theta - \cos \phi \tan \epsilon) \tan \alpha + \sin \phi \cos \theta)}{(\sin \phi \tan \epsilon + \cos \phi \sin \theta) \tan \alpha + \cos \phi \cos \theta}$$

Eliminating $\tan \alpha$ yields the equation for the sidereal hour-lines:

$$g_2 x \cos \theta \tan \epsilon - g_1 y (\cos \phi + \sin \phi \sin \theta \tan \epsilon) = g_1 g_2 (\cos \phi \sin \theta \tan \epsilon - \sin \phi) \qquad (14)$$

The equation[g] is linear in x and y; in the special case of $\phi = 90° - \epsilon$, these lines coincide with the Babylonian (for $\theta = b - 90°$) and Italian (for

$\theta = 270° - i$) hour-lines. Sidereal hour-lines for latitude 40°N under the assumptions $g_1 = \sec\phi$ and $g_2 = \tan\phi$ are drawn in figure 6.

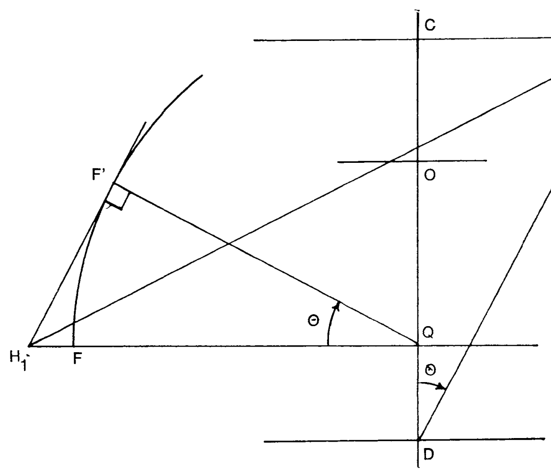

FIGURE 7.

A graphic construction of these lines similar to the one given for Babylonian hour-lines may be developed as follows[h]. In figure 7 let O be the origin and let points Q, C, D and F have the co-ordinates listed in Table VIII, where the special case is the one which generates figure 6. Construct lines CH_2 and QF perpendicular to QC. Let H_2 be the intersection of lines CH_2 and DH_2, where the angle $CDH_2 = \theta$. Let H_1 be the intersection of the line QF with the tangent to the circle (with centre Q and radius $\overline{QF}$) at point F′, where the angle $FQF' = \theta$. The sidereal hour-line for hour θ is H_1H_2. For the limiting cases when these points are at infinity, the hour-lines have the following equations:

$$\theta = 90° \rightarrow y = g_2 \tan(\phi - \epsilon)$$
$$\theta = 270° \rightarrow y = g_2 \tan(\phi + \epsilon)$$

TABLE VIII

CO-ORDINATES OF SELECTED POINTS FOR THE CONSTRUCTION OF SIDEREAL HOUR-LINES

Point	*General Case*		*Special Case*	
	X	Y	X	Y
Q	0	$-g_2 \cot\phi$	0	-1
C	0	$g_2 \tan\phi$	0	$\tan^2\phi$
D	0	$g_2 \tan\phi - g_1 \sec\phi$	0	-1
F	$-g_1 \operatorname{cosec}\phi \cot\epsilon$	$-g_2 \cot\phi$	$-\sec\phi \operatorname{cosec}\phi \cot\epsilon$	-1

To justify the construction, it suffices to determine that the points H_1 and H_2 satisfy equation (14).

$$H_1 = (-g_1 \operatorname{cosec}\phi \sec\theta \cot\epsilon,\ -g_2 \cot\phi)$$
$$H_2 = (g_1 \sec\phi \tan\theta,\ g_2 \tan\phi)$$

TABLE IX

HOUR-LINES OBTAINABLE FROM A SINGLE CONSTRUCTION

Line	*Sidereal Hour*
H_1H_2	θ
$H_1H'_2$	$360°-\theta$
H'_1H_2	$180°+\theta$
$H'_1H'_2$	$180°-\theta$

If we consider the points H_1' and H_2' as before, we see that each construction actually gives us four hour-lines (see Table IX).

Perhaps the primary interest of these hour-lines is that they demonstrate that a sundial is capable of recording time uniformly despite the fact that the apparent position of the Sun changes non-uniformly[i]. No correction needs to be made to the dial reading to obtain sidereal time. The value of θ at local noon on the vernal equinox is 0°; it continues to increase throughout the succeeding year, having a value of 90° on the summer solstice, 180° on the autumn equinox and 270° on the winter solstice.

Notes

(*a*) See ref. 10. Other references to bifilar dials include 1 (p. 105) and 2 (p. 135). Another brief geometrical treatment (ref. 5) of the basic equiangular dial appeared while the present article was in the hands of the referee.

(*b*) Dials of this type are discussed in ref. 11, where the treatment is not restricted to horizontal examples. This type of dial was invented in the seventeenth-century by Samuel Foster[3] but is often attributed to the eighteenth-century mathematician and philosopher J. H. Lambert[7]. The dial is discussed in ref. 1, and there have been a number of more recent studies of it, the most extensive being refs 4, 6 and 11.

(*c*) As a practical addition to the dial, one should consider some form of alidade or pointer free to rotate as a diameter of the dial face. If a line is drawn down the middle of the alidade, it can be turned until the intersection of the shadows lies on the line and the end of the alidade indicates the time on the circumference of the dial face. This addition would eliminate the need for drawing an excessive number of hour-lines. It should be noted that the heights of the threads must be measured from the upper surface of the alidade rather than from the surface of the dial proper.

Another practical point to make here is that if the threads are attached to the tops of their supports, the shadows of the supports will not obliterate any readings. However, if the supports extend above the threads, care should be taken to insure against problems caused by the shadows.

(*d*) The astronomical hour-lines for a dial at latitude ϕ with inclination i and declination d are given by the equation

$$g_1\, g_2 \cos \alpha = g_2 \cot (t-t') - g_1\, y \sin \alpha$$

where α and t' are given by equations (4) and (5).

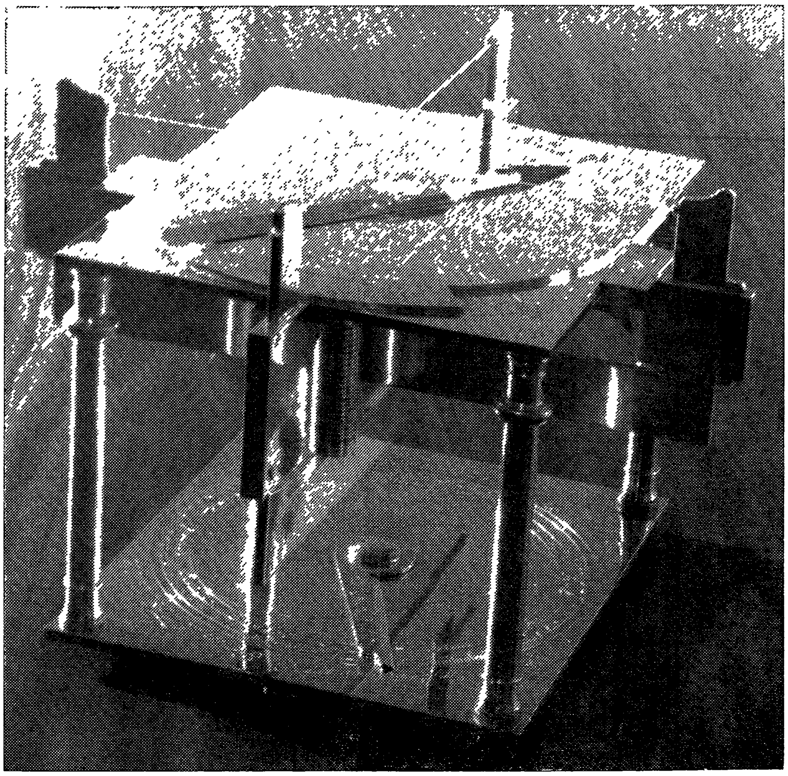

FIGURE 8. An equiangular bifilar sundial designed by M. U. Zakariya and the author. The dial is on the upper plate with hours marked around the circumference. The alidade has a line down its centre and must be turned until the intersection of the shadows of the threads lies on that line. The height and position of the threads are adjusted by micrometers on the underside of the plate, according to the latitude of the dial's location; adjustments for longitude and the equation of time are made with verniers on the dial face. The lower plate gives the equation of time. This dial is the first of a small series of bifilar dials to be constructed by Mr Zakariya.

(*e*) The day curve for a non-horizontal dial is

$$\sin^2 \delta \, (g_2^2 \, x^2 + g_1^2 \, y^2 + g_1^2 \, g_2^2) = g_1^2 \, (g_2 \sin \alpha - y \cos \alpha)^2$$

where α is given by equation (4).

(*f*) Michnik also considers temporary or unequal hour-lines, which result from dividing the period of daylight on any given day into 12 equal hours. The lines are given by higher-order algebraic equations (although they are often approximated by straight lines) and will not be considered here. They are treated in detail for the case of non-bifilar dials in ref. 8; the modification required to adapt them to the bifilar case is discussed in ref. 10.

(*g*) The equation for sidereal hour-lines for non-horizontal dials is

$$g_2 \, x \cos (\theta - t') \tan \epsilon - g_1 \, y \, (\cos \alpha + \sin \alpha \sin (\theta - t') \tan \epsilon) =$$
$$g_1 \, g_2 \, (\cos \alpha \sin (\theta - t') \tan \epsilon - \sin \alpha)$$

where α and t' are given by equations (4) and (5).

(*h*) For a somewhat different construction and development of sidereal hour-lines for non-bifilar sundials, see ref. 9.

(*i*) For the use to which such a sundial may be put in astrology, see ref. 1 (chapter XIV).

References

1 Drecker, J., *Die Theorie der Sonnenuhren*, Berlin, 1925.

2 Drecker, J., *Zeitmessung und Sterndeutung in geschichtlicher Darstellung*, Berlin, 1925.

3 Foster, S., *Elliptical, or azimuthal horologiography*, London, 1654.

4 Hanke, W., *Die Sterne*, **51,** 159 (1975).

5 Hanke, W., *Die Sterne*, **52,** 228 (1976).

6 Janin, L., *Le Cadran Solaire Analemmatique—Histoire et Développements.* Note 74.2057. Besançon, France: Centre Technique de l'Industrie Horlogère, 1974.

7 Lambert, J., *Berliner Astronomisches Jahrbuch für 1777*, 200. Berlin, 1775.

8 Michnik, H., *Beiträge zur Theorie der Sonnenuhren.* Leipzig, 1914.

9 Michnik, H., *Astronomische Nachrichten*, **216,** 441 (1922).

10 Michnik, H., *Astronomische Nachrichten*, **217,** 81 (1923).

11 Taylor, G., *J. Br. astron. Assoc.*, **86,** 7 (1975).